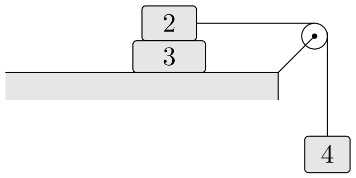
2001. Four cards are dealt in sequence from a standard deck. Find the probability that the first two cards are red and the second two black.

2002. (a) Express $8 \div 5$ as the sum of an integer and a proper fraction.

(b) Express $(2x+3) \div (x+1)$ as the sum of an integer and a proper algebraic fraction.

2003. A pulley system is set up on a table as depicted. Masses are given in kg. The pulley is light and smooth, and the string is light and inextensible. The coefficient of friction at both surfaces of the lower stacked block is $\frac{1}{2}$.



(a) Explain the modelling assumption that allows the system to be described with precisely two different accelerations.

(b) Show that these are zero and $\frac{1}{2}g$ ms^{-2}.

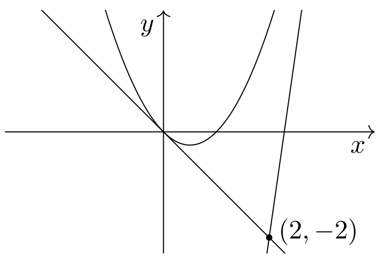
2004. Find the acute angle between the straight lines with equations $y = \sqrt{3}x + 1$ and $y = x$.

2005. Functions F and G are both periodic when defined over $\mathbb{R}$. F has period 4; G has period 6. Write down the periods of the following functions:

(a) $\text{F}(x)\,\text{G}(x)$,

(b) $\text{F}(2x) + \text{G}(3x)$.

2006. There are two tangents to the curve $y = x^2 - x$ which pass through the point $(2, -2)$. They meet the curve at the origin and at Q.



By using the discriminant, or by differentiation, find the coordinates of Q.

2007. Solve $\displaystyle\sum_{r=1}^{2} \frac{x^r}{1-x^r} = 0.$

2008. Two fair dice, one with m faces and one with n, where $m < n$, are rolled. Find, in terms of m and n, the probability that the scores are the same.

2009. Determine, in the form $\sqrt{a + b\sqrt{5}}$ for integers a, b, the exact value of $\sec 36°$, given that

$$\sin 36° = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.$$

2010. Prove that, if a is a positive constant, then the quartic graph $y = x^4 + ax^2 + b$ has at most one SP.

2011. Sketch $\sqrt{y} - \dfrac{1}{\sqrt{x}} = 0$.

2012. True or false?

(a) $y = x^2$ is a reflection of $x = y^2$,

(b) $y = -x^2 + 1$ is a rotation of $y = x^2$,

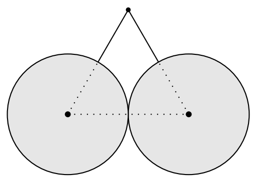
(c) $y = (x+1)^2$ is a translation of $y = (1-x)^2$.

2013. Let f be a linear function such that

$$\int_0^1 \text{f}(x)\,dx = 0 \quad \text{and} \quad \int_1^2 \text{f}(x)\,dx = 1.$$

Find f(2).

2014. Two Christmas baubles, modelled as identical spheres of radius r and mass m, are hung from the same point, each by a string of length r attached to its surface. The baubles hang in equilibrium.



Find the tension in the strings.

2015. The cosine function may be approximated, for small θ in radians, by $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Find the equivalent approximation for small θ in degrees.

2016. A projectile, launched at 15° above horizontal from flat ground, has range 40 m. Find the initial speed.

2017. Show that there is no real value of q for which $\frac{1}{9}(6\mathbf{i} + 8\mathbf{j} + q\mathbf{k})$ is of unit magnitude.

2018. The equation $acx^2 + (ad + bc)x + bd = 0$, where a, b, c, d are real constants, has exactly one real root. Show that $ad = bc$.

2019. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning real numbers x and y:

① $x^2 + y^2 = 1$,

② $x = \cos t, y = \sin t$ for some $t \in \mathbb{R}$.

2020. Prove that no quartic function is invertible over $\mathbb{R}$.

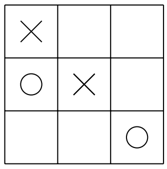
2021. The random variable X is distributed as $N(0, 1)$. Write down the distributions of

(a) $-X$,

(b) $1 - X$,

(c) $2X + 3$.

2022. In the game of tic-tac-toe, two players alternately place ○ and × in a grid. A player who places three in a row wins. In the game below, four moves have been played, with × to move next.



Prove that, if × plays logically, then ○ has lost.

2023. Solve the simultaneous equations

$$\begin{aligned} x + y + z &= 0, \\ xyz &= 2, \\ x(1 + y) &= 0. \end{aligned}$$

2024. Find $\displaystyle\int \frac{1}{3x - 4}\,dx$.

2025. Two dice have been rolled, giving scores X and Y. Determine whether the fact "$XY = 12$" increases, decreases or doesn't change $\mathbb{P}(X + Y = 7)$.

2026. A truck of mass 500 kg is pulling a trailer of mass 750 kg along a horizontal tarmac road, by means of a light tow-bar. Resistances 300 N and 450 N act on truck and trailer respectively. As the truck accelerates forwards from rest, the truck's wheels exert a frictional force of 1000 N on the tarmac.

(a) Draw force diagrams for truck and trailer.

(b) Calculate the tension in the tow-bar.

(c) The driver now kills the engine, and applies the trailer's brakes (not the truck's). Truck and trailer decelerate at 2 ms^{-2}. Show that this increases the tension in the tow-bar.

2027. Prove that $x^4 + x^2 + x^{-2} = 0$ is not satisfied by any real value of x.

2028. It can be shown that, for $\theta \in (0, \pi/2)$,

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Explain how this can be used to prove that, for small angles, $\sin\theta \approx \theta$.

2029. Simplify $\dfrac{\sqrt{p} + \sqrt{q}}{\sqrt{p} - \sqrt{q}} + \dfrac{\sqrt{p} - \sqrt{q}}{\sqrt{p} + \sqrt{q}}$.

2030. Find the following derivatives:

(a) $\frac{d}{dx}\left(x^2\right)$,

(b) $\frac{d}{dx}\left(y^2\right)$,

(c) $\frac{d}{dx}\left(e^{2y+1}\right)$.

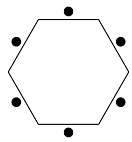
2031. A student writes: "Since the range of f, over the domain $\mathbb{R}$, is $(2, \infty)$, $\text{f}(x)$ has a local minimum with value 2." Explain the error, giving an explicit counterexample.

2032. A family of straight lines is defined by

$$y = \frac{1 + a}{1 - a}x.$$

Show that, if a can take any value in $\mathbb{R}$, this family contains all but one of the straight lines through the origin of the (x, y) plane.

2033. At a party, three couples sit down at random at a hexagonal table:



Find the probability that everyone ends up sitting next to their partner.

2034. The function $\text{f}(x) = px^2\left(q - x^2\right)$, defined over the domain $\mathbb{R}$, has a local maximum at $\text{f}(2) = 32$. Find the constants p and q.

2035. Solve $x^{20} - 1019x^{10} = 5120$.

2036. A person of mass 60 kg is standing on an accurate set of scales, which display a mass of 70 kg. But the person is carrying no extra mass. Give two different physical scenarios which would generate this behaviour.

2037. Four values x_1, x_2, x_3, x_4 are chosen at random and independently from the interval $[0, 1]$. Write down the probability that x_4 is the largest.

2038. Three vertices of a regular hexagon are given as $(2 \pm \sqrt{3}, 2)$, $(2, 3)$. Find the coordinates of the centre of the hexagon.

2039. Find the area scale factor when a region between the graph $y = a^x$ and the x axis is transformed to the equivalent region between $y = b^x$ and the x axis. Give your answer in simplified terms of a and b.

2040. A set of eighty data $x_i \in \mathbb{R}^+$ is summarised with a median m and an inter-quartile range $I > 0$. In further analysis, however, it turns out that the ten largest and ten smallest data were erroneously recorded, and are to be removed. Explain what effect rectifying this mistake will have on

(a) the sample median,

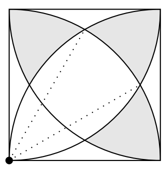
(b) the sample inter-quartile range.

2041. A curve is given, for $x \in (0, \infty)$, by

$$y = x - \frac{2}{\sqrt{x}}.$$

Show that the curve is increasing and concave.

2042. A pattern is constructed from a square and four quarter circles, as depicted below:



Show that, at the marked vertex, the three shaded regions all subtend the same angle.

2043. Verify that $\mathrm{f}(x) = \tan(x + k)$, for $k \in \mathbb{R}$, satisfies the differential equation $\mathrm{f}'(x) - (\mathrm{f}(x))^2 = 1$.

2044. A girl of mass 20 kg is standing on flat ground. She jumps over a puddle by exerting an extra force, in addition to the usual reaction of magnitude $20g$, of 300 Newtons on the ground, at 45° below the horizontal. This force is modelled as constant for the 0.2 s of takeoff.

(a) Draw a force diagram for the girl in takeoff.

(b) Show that she leaves the ground at 3 ms^{-1}, having moved 30 cm.

(c) Neglecting those 30 cm as a straightening of her legs, show that she leaps close to a metre.

2045. Prove that $\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})^2 \equiv \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$.

2046. A curve is given implicitly by $x^2y - xy^2 = 16$, and a line by $y = x + 4$.

(a) Solve to find the coordinates of any points of intersection between the curve and the line.

(b) Hence, by considering the nature of any roots, show that $y = x + 4$ is a tangent to the curve.

2047. Write the following in simplified interval notation:

$$\{x \in \mathbb{R} : |x - 4| \leq 6\} \setminus \{x \in \mathbb{R} : x \leq 1\}.$$

2048. Two functions f and g are such that $\mathrm{f}''(y) - \mathrm{g}''(y)$ is constant. Prove that the graphs $x = \mathrm{f}(y)$ and $x = \mathrm{g}(y)$ intersect at most twice.

2049. You are given that $y = k^x$. Write the following in terms of y:

(a) $k^{\frac{1}{2}x}$,

(b) k^a, where $a = \log_k 2 - \frac{1}{2}x$.

2050. Find a simplified expression for $\dfrac{d^2}{dx^2}(e^x \sin x)$.

2051. In a random experiment, events A and B are such that $\mathbb{P}(A) = 0.4$, $\mathbb{P}(B) = 0.2$, $\mathbb{P}(A \cap B) = 0.15$. Show that $\mathbb{P}(A \mid B) > \mathbb{P}(A' \mid B)$.

2052. Show that $\ln(1 + e^x)$ is increasing for all $x \in \mathbb{R}$.

2053. Provide a counterexample to the following claim: "No three primes are in arithmetic progression."

2054. You are given that the function f has roots at a and b, and that the function g has a root at b. Both are defined over the domain $\mathbb{R}$. For each of the following, either prove or disprove the statement.

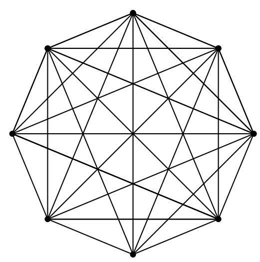
(a) "$\mathrm{f}(x) + \mathrm{g}(x) = 0$ must have a root at $x = b$."

(b) "$\mathrm{fg}(x) = 0$ must have a root at $x = b$."

(c) "$\mathrm{f}(x)\,\mathrm{g}(x) = 0$ must have a root at $x = a$."

2055. Solve $\left(\sqrt{x+1} + \sqrt{x}\right)^2 - \left(\sqrt{x+1} - \sqrt{x}\right)^2 = 0$.

2056. The diagram shows a regular octagon, which has 20 diagonals.



Prove that a regular n-sided polygon has $\frac{1}{2}n(n-3)$ diagonals.

2057. Prove that a polynomial graph of degree $n \geq 2$ can have a maximum of $n - 2$ points of inflection.

2058. Show that $\displaystyle\int_{-1}^{0} 1 - \frac{2x+1}{2x-1}\,dx = \ln 3$.

2059. Three distinct vertices are chosen at random from those of a regular hexagon. Find the probability that these vertices form an equilateral triangle.

2060. Show that $b_{n+1} = 3b_n^2 + 2b_n + 4$ has no fixed points.

2061. A bovine biologist is studying the rate of methane production in cows during gestation. For cows, the gestation period is taken to be 280 days following conception. Volume of methane is V litres. The rate of production is modelled as follows, with t as the number of days after conception:

$$\frac{dV}{dt} = 421 - 7.13 \times 10^{-4}t^2.$$

(a) Write down the baseline rate of production of a cow just prior to conception.

(b) Find the percentage change in rate of methane production over the gestation period.

(c) Find the percentage difference in total volume of methane produced for a pregnant cow as compared to a non-pregnant cow, over a 280 day gestation period.

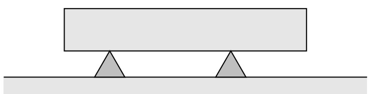
2062. Write $18x^2 + 72x - 4$ as a polynomial in $(3x + 2)$.

2063. It is given that xy is constant. Find the ratio

$$\left.\frac{dy}{dx}\right|_{x=3} : \left.\frac{dy}{dx}\right|_{x=1}$$

2064. A sample $\{x_i\}$ of size n has mean $\bar{x}$ and variance s^2. In terms of these quantities, find the mean value of $x_i^2 - \bar{x}^2$.

2065. A uniform block of mass m kg rests on supports, as depicted. The supports divide the length of the block into sections in the ratio $a : 1 : b$.



Show that the magnitudes of the reaction forces at the supports are given by

$$R = \tfrac{1}{2}mg\big(1 \pm (a - b)\big).$$

2066. Prove that, if the diagonals of a parallelogram are perpendicular, then the shape must be a rhombus.

2067. A fair coin is tossed four times, and the number of tosses in the longest run of either heads or tails is recorded as X. Find $\mathbb{P}(X = 2)$.

2068. Curve C is defined by $y = \left(\frac{1}{3}x + 2\right)^3 - \left(\frac{1}{3}x + 2\right)$.

(a) Determine the x-axis intercepts of C.

(b) Show that C is stationary at $x = a \pm \sqrt{b}$, where a and b are integers to be determined.

(c) Sketch the curve C, labelling axis intercepts.

2069. Prove that no number of the form $4p^2 + 1$, where p is a prime, is a perfect square.

2070. By sketching the boundary equations, show that no (x, y) points satisfy $y < 2 - 3\,|x - 2|$ and $y > x^2$.

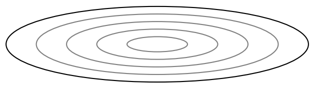
2071. For a board game, a pair of special six-sided dice are rolled. Each has two white faces, two black faces, and two red faces. Find the probability that

(a) no red faces show,

(b) one black and one white face show,

(c) one black and one white face show, given that no red faces show.

2072. A circular ripple is spreading across a pond. Its radius is increasing linearly with time, at a rate of 5 centimetres per second.



Find the rate of change of the area of the circle, when the radius is 10 cm.

2073. A cube has opposite faces $ABCD$ and $EFGH$, where AE is an edge. Find, in the form $\arcsin k$, the exact values of

(a) $\angle ABE$,

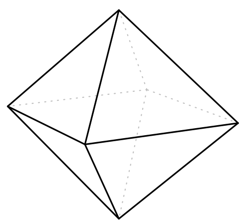
(b) $\angle ACE$.

2074. Without a calculator, evaluate $\displaystyle\sum_{k=1}^{3} \operatorname{cosec}^2 \frac{\pi k}{4}$.

2075. The curve $y = \sqrt{x} - x$ has tangents drawn to it at both of its x intercepts. Show that these tangents meet at the point $(0, 1/2)$.

2076. Solve $(x - 1)^3(x + 1)^2 + (x^2 - 1)^2 = 0$.

2077. A regular octahedron is shown below.



Four of the faces of the octahedron are selected at random and coloured. Find the probability that no coloured faces share an edge.

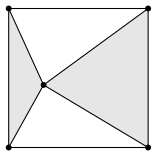
2078. A student of mathematics is attempting to find the equation of the tangent to $y = x^2 + x$ at the origin. He writes: “The gradient is given by $\frac{dy}{dx} = 2x + 1$. So, in $y = mx + c$, we have $y = (2x + 1)x + c$. The line goes through the origin, so $c = 0$. Hence, the line is $y = (2x + 1)x$.” Explain the error in his calculations, and correct it.

2079. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$y > x, \qquad x^2 + y^2 < 4.$$

2080. If $t = \tan u$, show that $\frac{du}{dt} = 1 - \sin^2 u$.

2081. From a point inside a square, four triangles are set up, as shown. Two opposite triangles are shaded.



Prove that half of the square is shaded.

2082. Simplify $\frac{d}{dt}\left(x^2 + y^2\right) + \frac{d}{dt}\left(x^2 - y^2\right)$.

2083. A sample $\{x_i\}$, mean $\bar{x}$ and standard deviation s_x, is transformed to give a new sample $\{y_i\}$, using the formula $y_i = x_i^2$.

(a) Show that $\bar{y} = \bar{x}^2 + s_x^2$.

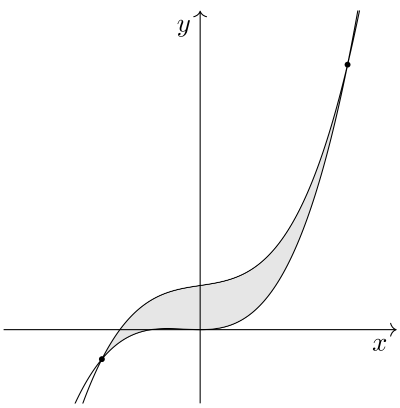
(b) Explain why it is not possible to find s_y with the information given.

2084. Prove, from first principles, that differentiation is linear, i.e. that, for constants $a, b \in \mathbb{R}$,

$$\big(a\,\mathrm{f}(x) + b\,\mathrm{g}(x)\big)' = a\,\mathrm{f}'(x) + b\,\mathrm{g}'(x).$$

2085. Solve $\displaystyle\sum_{r=1}^{3} \frac{1}{x - r} = 0$.

2086. The diagram shows the curves $y = x^3 + x^2$ and $y = x^3 + x + 6$, and a region enclosed by them.



Show that the area of the shaded region is $\frac{125}{6}$.

2087. Prove, by contradiction, that there is no smallest positive rational number.

2088. Giving your answer to 3sf, solve the equation

$$\left(2^x + 6\right)\left(3^{2x} + 3^x - 5\right) = 0.$$

2089. The cubic function $\mathrm{f}(x) = x^3 - 9x^2 + 26x + k$ has three integer roots.

(a) Find the x coordinates of the two stationary points of $y = \mathrm{f}'(x)$, giving your answers to 3sf.

(b) Hence, find the value of k.

2090. One of the following statements is true; the other is not. Identify and disprove the false statement.

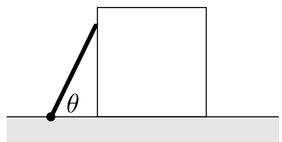
① $\cos\beta = 1 \implies \sin\beta = 0$,

② $\cos\beta = 0 \impliedby \sin\beta = 0$.

2091. An arithmetic sequence has first term 4, common difference 3, and there are 40 terms. A term is chosen at random. Find the probability that this term is a three-digit number.

2092. You are given that $9a^2b^{-\frac{1}{2}} + 4b^{\frac{1}{2}} = 12a$, and that $a, b > 0$. Write a in simplified terms of b.

2093. A mechanical switch inside a machine is built as follows: a uniform rod of mass m is freely hinged to a horizontal surface, and leans up against a fixed block, in equilibrium. The contact between rod and block is modelled as smooth.



(a) Draw a force diagram for the rod.

(b) Show that, at the hinge, the rod experiences a contact force of magnitude

$$\tfrac{1}{2}mg\sqrt{\operatorname{cosec}^2\theta + 3}.$$

2094. State, with a reason, whether the following holds: "In a one-tail binomial hypothesis test, if

$$\mathbb{P}(X = k) < \tfrac{1}{20},$$

then, at the 5% level, k lies in the critical region for the test."

2095. Function F is defined by

$$\mathrm{F}(x) = \int_0^x \mathrm{f}(t)\,dt,$$

where f is a polynomial function defined over the domain $[0, \infty)$. State, with a reason in each case, whether the following are necessarily true:

(a) $\mathrm{F}(0) = 0$,

(b) $\mathrm{F}(ax + b) \equiv a\,\mathrm{F}(x) + b$, for constants a, b,

(c) $\mathrm{F}'(x) \equiv \mathrm{f}(x)$.

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2096. Find the equation of the tangent to $y = e^{\sin 2x}$ at the curve's y intercept.

2097. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning real numbers x and y:

① $|x| = |y|$,

② $x^3 = y^3$.

2098. The area of the region enclosed by the curves $y = x^2$ and $y = kx - x^2$ is $\frac{8}{3}$. Find the constant k.

2099. (a) Find the prime factors of 219.

(b) Find integers a and b in the following:

$$\left(a\sqrt{2} + b\sqrt{3}\right)^3 = 486\sqrt{2} + 219\sqrt{3}.$$

2100. A cubic function is defined as

$$\mathrm{f}(x) = 3x^3 - 11x^2 - 11x - 14.$$

(a) Find $\mathrm{f}'(x)$.

(b) Set up the Newton-Raphson iteration, and use it to find the rational root of the equation.

(c) Hence, factorise and solve fully.

——— End of 21st Hundred ———